Relations between Kauffman and Homfly satellite invariants

H. R. Morton and N. D. A. Ryder

Department of Mathematical Sciences
University of Liverpool
Peach Street, Liverpool L69 7ZL

Abstract

We extend a mod 2 relation between the Kauffman and Homfly polynomials, first observed by Rudolph in 1987, to the general Kauffman and Homfly satellite invariants.

1   Introduction

The result of Rudolph [5] relating the Kauffman polynomial of a link to the Homfly polynomials of reverse parallels of its sublinks has long provided an intriguing and rather isolated connection betwewen these two major knot invariants. In this paper we show how Rudolph's result can be extended to general Kauffman satellite invariants.

Our results connect the meridian eigenvectors y_λ of the Kauffman skein of the annulus with the orientation-symmetric eigenvectors R_λ of the Homfly skein in a combinatorially suggestive way, which begs for a representation-theoretic explanation. The proofs, however, rely purely on the representation-related combinatorics of these eigenvectors, and features of their eigenvalues, while depending eventually on Rudolph's original skein-theoretic result as a basis for the inductive proof.

1.1   Satellite invariants and skeins of the annulus

We use the framed version of the Homfly polynomial based on the skein relations

and normalised to take the value 1 on the empty link.

Similarly the defining skein relations for the Kauffman polynomial are

Throughout the paper coefficients are taken in the ring $\Lambda = \mathbf{Z}_2[v^{\pm 1}, s^{\pm 1}]$ with the elements $\{s^r - s^{-r}, r \in \mathbf{Z}_{>0}\}$ inverted. As in Rudolph's original result the use of $\mathbf{Z}_2$ rather than $\mathbf{Z}$ is essential.

A *satellite* of a framed knot K is determined by choosing a diagram Q in the standard annulus, and then drawing Q on the annular neighbourhood of K determined by the framing. This gives the satellite of K when *decorated* by the diagram Q. The result is the satellite knot denoted by $K * Q$.
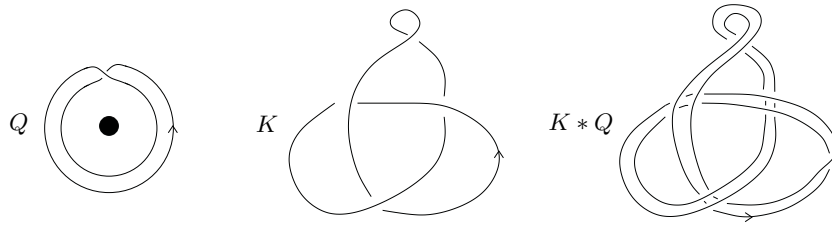
Starting with a framed link

$$L = \cup_{i=1}^k L_i$$

we may decorate each component with a diagram Q_i in the annulus. The Kauffman or Homfly polynomial of the resulting link, is called a *satellite invariant* of L, as detailed for example in [3] in the case of the Homfly polynomial. We write $D(L; Q_1, \ldots, Q_k)$ and $P(L; Q_1, \ldots, Q_k)$ for the Kauffman and Homfly polynomials respectively of the link L when each component L_i is decorated by Q_i.

The satellite invariants depend only on the decorations Q_i as elements in the appropriate skein of the annulus (consisting of Λ-linear combinations of diagrams in the annulus, modulo the corresponding skein relations). The general linear combination of satellite invariants can be determined in terms of the satellite invariants where the decorations are drawn from a linear basis of the skein of the annulus.

1.2 The Kauffman skein

The Kauffman skein of the annulus has a basis $\{y_\lambda\}$, described by Blanchet and Beliakova [1], where λ runs through the set of all partitions. These elements consist of eigenvectors of the meridian map, [7], and are closely related to the irreducible modules used in constructing link invariants from the B, C and D series of quantum groups.

When λ is a partition of 1 we write $y_\lambda = y_1$. This decoration can be represented in the skein of the annulus by the single core curve of the annulus. Then $D(L; y_1, \ldots, y_1)$ is just the Kauffman polynomial of the link L itself, with coefficients reduced mod 2.

1.3 The Homfly skein

The Homfly skein of the annulus has a basis $\{Q_{\lambda,\mu}\}$, described by Hadji and Morton [3], where λ and μ run through the set of all partitions, including the empty partition ϕ. Diagrams in the Homfly skein are oriented. Reversing the orientation of all components in a diagram induces an involution on the skein which carries the basis element $Q_{\lambda,\mu}$ to $Q_{\mu,\lambda}$.

When λ is a partition of 1 and μ is empty the decoration $Q_{1,\phi}$ can be represented by a single oriented core curve, while $Q_{\phi,1}$ is represented by the oppositely oriented core curve, and $Q_{\phi,\phi}$ by the empty diagram.

The result presented here in Theorem 1 shows how the Kauffman invariant with decoration y_λ on a component of L relates to the Homfly invariant with decoration $R_\lambda = Q_{\lambda,\lambda}$ on the corresponding component.

The decoration $R_1 = Q_{1,1}$ for Homfly can be termed the *adjoint decoration*, because of its relation with the adjoint representation for the A series of quantum groups. In terms of the product in the skein of the annulus induced by placing decorations in concentric annuli we can write $Q_{1,1} = Q_{1,\phi}Q_{\phi,1} - Q_{\phi,\phi}$, which is realised as the decoration by two parallel oppositely oriented core curves minus the empty decoration. The *adjoint Homfly polynomial* $P_{\mathrm{adj}}(L)$ is given by using the adjoint decoration R_1 on each component of an unoriented link L.

1.4 Rudolph's relation and its extension

Squaring in the ring Λ is a ring homomorphism which we write as $\overline{} : \Lambda \to \Lambda$.

Rudolph's original result [5] says that the adjoint Homfly polynomial of a link L is equal to its Kauffman polynomial in the ring Λ after the substitution $v \to v^2, s \to s^2$. This can be summarised as

$$P_{\mathrm{adj}}(L) = \overline{D(L)}.$$

We extend Rudolph's result to general satellite invariants as follows.

Theorem 1. *For any choice of partitions $\lambda(1), \ldots, \lambda(k)$ the Homfly satellite invariant $P(L; R_{\lambda(1)}, \ldots, R_{\lambda(k)})$ of the link L satisfies*

$$
\begin{aligned}
P(L; R_{\lambda(1)}, \ldots, R_{\lambda(k)}) &= \overline{D(L; y_{\lambda(1)}, \ldots, y_{\lambda(k)})} \\
&= D(L; y_{\lambda(1)}, \ldots, y_{\lambda(k)})|_{v \to v^2, s \to s^2}.
\end{aligned}
$$

1.5 The strategy for a proof of Theorem 1

The proof depends on the very similar behaviour of the Kauffman and Homfly skeins of the annulus when a further meridian or longitude is adjoined. The effect is determined by the meridian maps, established by Lu and Zhong [7] for Kauffman and by Hadji and Morton [3] for Homfly, and the branching rules, described by Beliakova and Blanchet [1] for Kauffman and by Hadji [2] for Homfly.

It is possible to produce a linear combination $\sum a_j L^{(j)}$ of links $L^{(j)}$, each consisting of a succession of many longitude and meridian curves around each component L_i of the original link L, with the property that

$$D(L; y_{\lambda(1)}, \ldots, y_{\lambda(k)}) = \sum a_j D(L^{(j)})$$

and

$$P(L; R_{\lambda(1)}, \ldots, R_{\lambda(k)}) = \sum \overline{a_j} P_{\mathrm{adj}}(L^{(j)}).$$

Rudolph's result applied to $L^{(j)}$ would then establish the theorem.

Here we prove Theorem 1 by induction on $\sum |\lambda(i) - 1|$, with Rudolph's result as the base, using an intermediate version of the longitude and meridian technique.

2 Meridian maps

The basic information needed about the skeins of the annulus are firstly the behaviour of the meridian maps.

2.1 Kauffman meridian maps

In the Kauffman skein Lu and Zhong [7] show that when y_1 is placed on a meridian around y_λ the result is $c_\lambda y_\lambda$, and the scalars c_λ for different partitions are all different.
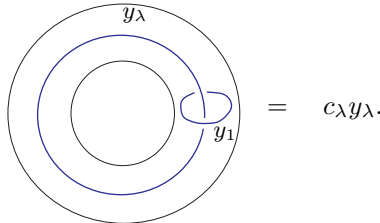
The eigenvalue c_λ calculated by Lu and Zhong uses coefficients in $\mathbf{Z}[v^{\pm 1}, s^{\pm 1}]$ with denominators as in Λ.

Theorem 2 (Lu and Zhong).

$$c_\lambda = (s - s^{-1}) \left(v^{-1} \sum_{x \in \lambda} s^{2c(x)} - v \sum_{x \in \lambda} s^{-2c(x)} \right) + \frac{v^{-1} - v}{s - s^{-1}} + 1.$$

Here x runs through cells in a diagram of the partition λ, and $c(x) = j - i$ denotes the content of the cell x in position (i, j).

The result of Lu and Zhong that these eigenvalues are distinct holds even when they are evaluated with mod 2 coefficients in the ring Λ.

Theorem 3. *If $c_\lambda = c_\mu$ when evaluated in Λ then $\lambda = \mu$.*

Proof. The *content polynomial* $C_\lambda(t)$ of a partition λ is the integer Laurent polynomial

$$C_\lambda(t) = \sum_{x \in \lambda} t^{c(x)}.$$

As above, $c(x)$ denotes the content of the cell x in a Young diagram of λ. The eigenvalue c_λ can then be written as

$$c_\lambda = (s - s^{-1}) \left(v^{-1} C_\lambda(s^2) - v C_\lambda(s^{-2}) \right) + \frac{v^{-1} - v}{s - s^{-1}} + 1.$$

Write the partition λ in Frobenius form as $(a_1, \ldots, a_k | b_1, \ldots, b_k)$, having arms of lengths $a_1 > a_2 > \ldots a_k \geq 0$ and legs of lengths $b_1 > b_2 > \ldots b_k \geq 0$. Then

$$(s - s^{-1}) C_\lambda(s^2) = \sum_{i=1}^{k} (s^{2a_i + 1} - s^{-2b_i - 1}),$$

since the terms in the content polynomial $C_\lambda(s^2)$ coming from the cells in the ith arm and leg of λ are $\displaystyle\sum_{r=-b_i}^{a_i} s^{2r}$.

Then $(s - s^{-1}) C_\lambda(s^2)$ determines the Frobenius form of λ, even when given as a Laurent polynomial in s with $\mathbf{Z}_2$ coefficients, as there can be no cancellation among the $2k$ terms.

The coefficient of v^{-1} in $c_\lambda - c_\mu$ is $(s - s^{-1}) C_\lambda(s^2) - (s - s^{-1}) C_\mu(s^2)$. If $c_\lambda = c_\mu$ in Λ then $(s - s^{-1}) C_\lambda(s^2) = (s - s^{-1}) C_\mu(s^2)$ in $\mathbf{Z}_2[s^{\pm 1}]$, and so $\lambda = \mu$. ∎

2.2 Homfly meridian maps

In the Homfly skein Hadji and Morton [3] show that the basis element $Q_{\lambda,\mu}$ is an eigenvector for the meridian map on the Homfly skein of the annulus induced by placing an oriented meridian around the annulus.

$$= \quad s_{\lambda,\mu} Q_{\lambda,\mu}.$$
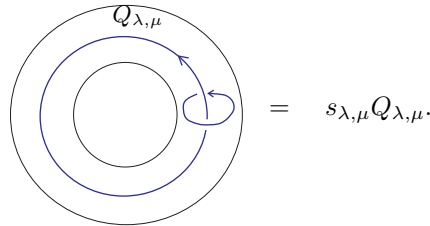
Theorem 4 (Hadji and Morton)**.** *The eigenvalue $s_{\lambda,\mu}$ is given by the formula*

$$s_{\lambda,\mu} = (s - s^{-1}) \left(v^{-1} C_\lambda(s^2) - v C_\mu(s^{-2}) \right) + \frac{v^{-1} - v}{s - s^{-1}}.$$

Orienting the meridian in the opposite sense results in the eigenvalue $s_{\mu,\lambda}$.

Comparing the formulae in Theorems 2 and 4 shows immediately that $c_\lambda = s_{\lambda,\lambda} + 1$.

Theorem 5. *When the adjoint decoration R_1 is placed on a meridian around $Q_{\lambda,\mu}$ the resulting element in the Homfly skein is $(s_{\lambda,\mu} s_{\mu,\lambda} - 1) Q_{\lambda,\mu}$.*

Proof. Since $R_1 = Q_{1,\phi} Q_{\phi,1} - 1$ we can use the eigenvalue results above. ∎

Theorem 6. *When the adjoint decoration R_1 is placed on a meridian around R_λ the resulting element in the Homfly skein using the ring Λ is $\overline{c_\lambda} R_\lambda$.*

$$= \quad \overline{c_\lambda} R_\lambda.$$

Proof. Since $R_\lambda = Q_{\lambda,\lambda}$ we have

$$= \quad (s_{\lambda,\lambda}^2 - 1) R_\lambda.$$

In the ring Λ we have $\overline{c_\lambda} = c_\lambda^2 = s_{\lambda,\lambda}^2 + 1 = s_{\lambda,\lambda}^2 - 1$. ∎

3 Branching rules

The skeins of the annulus admits a commutative product, induced by placing two skein elements on parallel strands. When a single parallel strand is added in the skein the description of the resulting product is determined by the branching rules.

3.1 Kauffman branching rules

In the Kauffman skein of the annulus two parallel strands, one decorated by y_ρ and the other by y_1 as in Figure 1, can be written as a sum of basis elements using the branching rules of Beliakova and Blanchet [1].
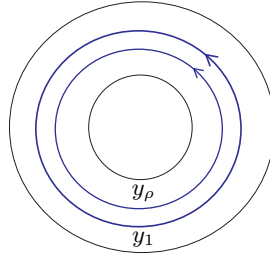

Figure 1: The product $y_\rho y_1$

The branching formula for the product $y_\rho y_1$ is given in the following theorem.

Theorem 7 (Beliakova and Blanchet)**.**

$$y_\rho y_1 = \sum_{\mu \in \rho^+ \cup \rho^-} y_\mu,$$

where ρ^+ and ρ^- are the sets of partitions given from ρ by respectively adding or deleting one cell.

3.2 Homfly branching rules

A single oriented core curve in the Homfly skein of the annulus represents $Q_{1,\phi}$ or $Q_{\phi,1}$ depending on its direction.

The basic branching rules in the Homfly skein for two parallel strands, one decorated by $Q_{\alpha,\beta}$ and one by $Q_{1,\phi}$, as in Figure 2, are

$$Q_{\alpha,\beta} Q_{1,\phi} = \sum_{\mu \in \alpha^+} Q_{\mu,\beta} + \sum_{\nu \in \beta^-} Q_{\alpha,\nu},$$

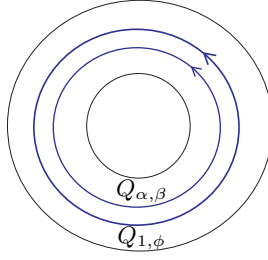while reversing the direction in the annulus gives a similar formula for the product with $Q_{\phi,1}$.

Figure 2: The product $Q_{\alpha,\beta}Q_{1,\phi}$

We can then expand the product $R_\rho R_1 = Q_{\rho,\rho}(Q_{1,\phi}Q_{\phi,1} - 1)$ as follows.

Theorem 8.

$$R_\rho R_1 = \sum_{\mu \in \rho^+ \cup \rho^-} R_\mu + \sum_{\substack{\{\alpha,\beta\} \\ \alpha \neq \beta}} n_{\alpha,\beta}(Q_{\alpha,\beta} + Q_{\beta,\alpha}) + 2|\rho^-|R_\rho,$$

for some integers $n_{\alpha,\beta}$.

The main importance of this expansion for us is its similarity to the expansion for $y_\rho y_1$ in the Kauffman skein and the fact that the remaining terms occur in pairs $Q_{\alpha,\beta} + Q_{\beta,\alpha}$. There is an explicit calculation in [6] of the coefficients $n_{\alpha,\beta}$, which are either 1 or 0, but the details are not needed here.

4 Longitude-meridian decorations

Using the branching rules and the meridian map it is possible to express any element in the Kauffman skein of the annulus as a Λ-linear combination of *longitude-meridian* diagrams.

We define a longitude-meridian diagram recursively, starting from a single core curve, by including a sequence of further longitude and meridian curves. These can be specified by a sequence of powers of l and m, as in Figure 3.
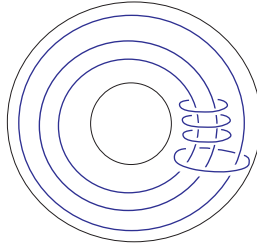

Figure 3: The longitude-meridian diagram $l^2 m^3 l m$

For a partition λ with $|\lambda| > 1$ choose a subpartition ρ with one fewer cell. We can reproduce the effect of the single element y_λ in the Kauffman skein of the annulus in terms of a linear combination of longitude-meridian diagrams in which each component is decorated by y_1 except for one innermost longitude decorated by y_ρ as follows.

Write

$$X(t) = \prod_{\mu \in \rho^+ \cup \rho^- - \{\lambda\}} (t - c_\mu) \tag{1}$$

as a polynomial $\sum a_r t^r$ in t.

The longitude-meridian diagram $l^2 m^r$ decorated with y_ρ on one longitude and with y_1 on the other curves is the product $y_\rho y_1$ surrounded by r meridians.

Since $y_\rho y_1 = \sum_{\mu \in \rho^+ \cup \rho^-} y_\mu$ this decorated longitude-meridian diagram represents $\sum_{\mu \in \rho^+ \cup \rho^-} c_\mu^r y_\mu$ in the Kauffman skein of the annulus.

Then the linear combination $\sum_r a_r l^2 m^r$ of longitude-meridian diagrams decorated with y_ρ on one longitude and with y_1 elsewhere represents

$$\sum_r a_r \left(\sum_{\mu \in \rho^+ \cup \rho^-} c_\mu^r y_\mu \right) = \sum_{\mu \in \rho^+ \cup \rho^-} X(c_\mu) y_\mu$$
$$= X(c_\lambda) y_\lambda, \text{ by (1)}.$$

Since $X(c_\lambda) \neq 0$ in Λ, by Theorem 3, this represents y_λ up to a non-zero multiple.

For example, the decoration for $r = 3$ is shown in Figure 4.
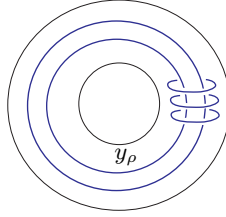

Figure 4: The decoration for the coefficient of t^3 in $X(t)$.

Although we will not need to expand further it is clear that the longitude decorated by y_ρ could itself be replaced recursively by a linear combination of longitude-meridian diagrams, to give a representation of y_λ in the skein entirely in terms of longitude-meridian diagrams.

5 The proof of Theorem 1

We proceed by induction on $N = \sum(|\lambda(i)| - 1)$. Components which are assigned the partition 1 do not then contribute to N. The base case with $N = 0$, where all components have the partition 1, is given by Rudolph's result.

If $N > 0$ then $|\lambda(i)| > 1$ for some i. We will suppose that $i = 1$ and write $\lambda = \lambda(1)$.

Write $L^{(r)}$ for the link consisting of two parallel copies of L_1 surrounded by r meridians, along with the remaining components of L. This is the satellite of L in which the component L_1 is decorated by the longitude-meridian diagram $l^2 m^r$.

Choose a partition $\rho \in \lambda^-$. This is possible since $|\lambda| > 1$. Decorate $L^{(r)}$ with y_ρ on one parallel of L_1, y_1 on the other parallel and on the r meridians, and with the original decorations $y_{\lambda(j)}$ on $L_2, \ldots, L_k$, and write $D^{(r)}$ for its Kauffman invariant.

Write $P^{(r)}$ for the Homfly polynomial of $L^{(r)}$ when decorated by the corresponding elements R_ρ, R_1 and $R_{\lambda(j)}$.

The element in the Homfly skein represented by the diagram $l^2 m^r$ decorated by R_ρ on one longitude and R_1 on all the other curves represents

$$\sum_{\mu \in \rho^+ \cup \rho^-} (s_{\mu,\mu}^2 - 1)^r R_\mu + \sum_{\substack{\{\alpha,\beta\} \\ \alpha \neq \beta}} n_{\alpha,\beta}(s_{\alpha,\beta} s_{\beta,\alpha} - 1)^r (Q_{\alpha,\beta} + Q_{\beta,\alpha}) + 2|\rho^-|(s_{\rho,\rho}^2 - 1)^r R_\rho.$$

Hence we can write $P^{(r)}$ as the sum of invariants

$$
\begin{aligned}
P^{(r)} \quad = \quad & \sum_{\mu \in \rho^+ \cup \rho^-} (s_{\mu,\mu}^2 - 1)^r P(L; R_\mu, R_{\lambda(2)}, \ldots, R_{\lambda(k)}) \\
& + \sum_{\substack{\{\alpha,\beta\} \\ \alpha \neq \beta}} n_{\alpha,\beta}(s_{\alpha,\beta} s_{\beta,\alpha} - 1)^r (P(L; Q_{\alpha,\beta}, R_{\lambda(2)}, \ldots, R_{\lambda(k)}) \\
& + P(L; Q_{\beta,\alpha}, R_{\lambda(2)}, \ldots, R_{\lambda(k)})) \\
& + 2|\rho^-|(s_{\rho,\rho}^2 - 1)^r P(L; R_\rho, R_{\lambda(2)}, \ldots, R_{\lambda(k)}).
\end{aligned}
$$

Since the Homfly polynomial of a link is unchanged when the orientation of *all* its components are reversed we have

$$
P(L; Q_{\alpha,\beta}, R_{\lambda(2)}, \ldots, R_{\lambda(k)}) = P(L; Q_{\beta,\alpha}, R_{\lambda(2)}, \ldots, R_{\lambda(k)}).
$$

Then in Λ we have the simplified expression

$$
P^{(r)} = \sum_{\mu \in \rho^+ \cup \rho^-} (\overline{c_\mu})^r P(L; R_\mu, R_{\lambda(2)}, \ldots, R_{\lambda(k)}), \tag{2}
$$

since $s_{\mu,\mu}^2 - 1 = \overline{c_\mu}$ and the remaining terms cancel mod 2.

The induction count N has reduced by 1 for the choice of partitions for the links $L^{(r)}$ when compared with the choice for the original link L.

Then $P^{(r)} = \overline{D^{(r)}}$, by induction.

Define the polynomial

$$
X(t) = \sum a_r t^r = \prod_{\mu \in \rho^+ \cup \rho^- - \{\lambda\}} (t - c_\mu),
$$

as in equation (1). Then

$$
\sum a_r D^{(r)} = X(c_\lambda) D(L; y_\lambda, y_{\lambda(2)}, \ldots, y_{\lambda(k)}).
$$

Now

$$
\begin{aligned}
\sum \overline{a_r D^{(r)}} \quad & = \quad \sum \overline{a_r}\, \overline{D^{(r)}} \\
& = \quad \sum \overline{a_r} P^{(r)} \\
& = \quad \sum_{\mu \in \rho^+ \cup \rho^-} (\sum \overline{a_r}(\overline{c_\mu})^r) P(L; R_\mu, R_{\lambda(2)}, \ldots, R_{\lambda(k)}), \text{ by (2)}, \\
& = \quad \sum_{\mu \in \rho^+ \cup \rho^-} \overline{X(c_\mu)} P(L; R_\mu, R_{\lambda(2)}, \ldots, R_{\lambda(k)}) \\
& = \quad \overline{X(c_\lambda)} P(L; R_\lambda, R_{\lambda(2)}, \ldots, R_{\lambda(k)}), \text{ by (1)}.
\end{aligned}
$$

Then $\overline{X(c_\lambda)}\; \overline{D(L; y_\lambda, y_{\lambda(2)}, \ldots, y_{\lambda(k)})} = \overline{X(c_\lambda)} P(L; R_\lambda, R_{\lambda(2)}, \ldots, R_{\lambda(k)})$.

Theorem 1 follows immediately, since $X(c_\lambda) \neq 0$ in Λ, and Λ has no zero-divisors.

Acknowledgments

References

[1] A Beliakova and C Blanchet. Skein construction of idempotents in Birman-Murakami-Wenzl algebras. *Math. Ann.* 321 (2001), 347–373.

[2] RJ Hadji. Homfly skein theory of reversed string satellites. PhD thesis, University of Liverpool 2003.

[3] RJ Hadji and HR Morton. A basis for the full Homfly skein of the annulus. *Math. Proc. Cambridge Philos. Soc.* 141 (2006), 81–100.

[4] HR Morton. Integrality of Homfly 1-tangle invariants. *Algebr. Geom. Topol.* 7 (2007), 327–338.

[5] L Rudolph. A congruence between link polynomials. *Math. Proc. Cambridge Philos. Soc.* 107 (1990), 319–327.

[6] NDA Ryder. Skein based invariants and the Kauffman polynomial. PhD thesis, University of Liverpool, 2008.

[7] B Lu and JK Zhong. The Kauffman polynomials of generalized Hopf links. *J. Knot Theory Ramifications* 11 (2002), 1291–1306.